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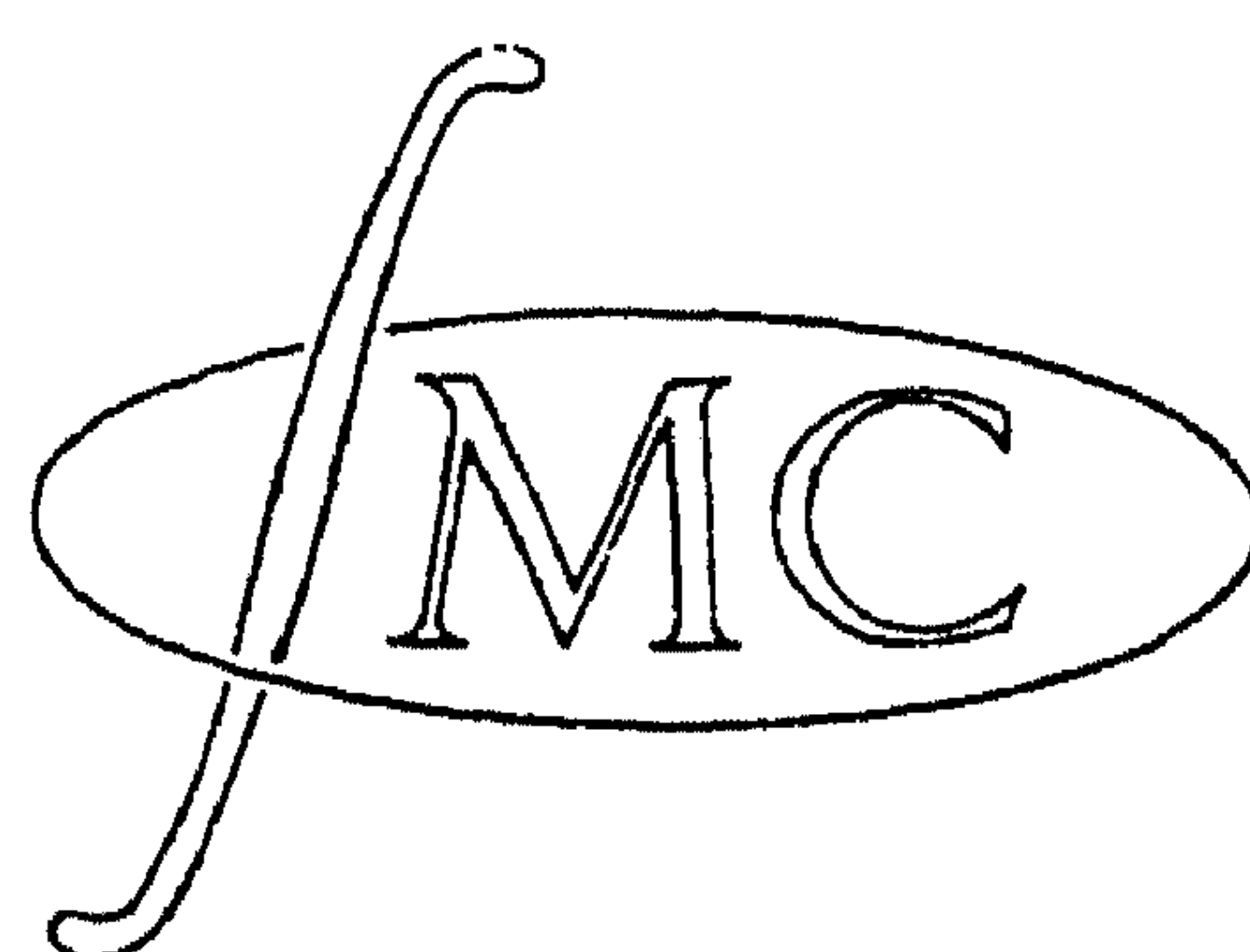
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On the connection between the arithmetico-geometrical  
mean and the complete elliptic integral of the first kind

by

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## Introduction

Subject of this note is the relation between the arithmetico-geometrical mean and the complete elliptic integral of the first kind.

Let  $a$  and  $b$  be two positive numbers, let the series  $\{a_n\}$  and  $\{b_n\}$  be defined by the recurrent relations:

$$(1) \quad a_{n+1} = \frac{a_n + b_n}{2} \quad \text{and} \quad b_{n+1} = \sqrt{a_n b_n},$$

with  $a_0 = a$ ,  $b_0 = b$ ;

without loss generality it may be assumed that  $a \geq b$ .

As can be easily seen, both series converge to the same limit, denoted by  $M(a, b)$  and called the arithmetico-geometrical mean of  $a$  and  $b$ .

The complete elliptic integral of the first kind is defined as:

$$(2) \quad K(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 \phi)^{-\frac{1}{2}} d\phi.$$

In the first section we derive the relation

$$(3) \quad \int_0^{\pi/2} (a_0^2 \sin^2 \phi + b_0^2 \cos^2 \phi)^{-\frac{1}{2}} d\phi = \int_0^{\pi/2} (a_1^2 \sin^2 \phi + b_1^2 \cos^2 \phi)^{-\frac{1}{2}} d\phi$$

by aid of a possibly new method. This method is based on potential theory and is due to the late prof. B. van der Pol. It may be remarked, that this relation is also a direct consequence of Landen's transformation, applied to the left-hand side of (3).

By aid of (3) we can easily establish the relation between the arithmetico-geometrical mean and the complete elliptic integral of the first kind, viz.

$$(4) \quad \frac{\pi}{2M(a, b)} = \frac{1}{a} K\left(\sqrt{1 - \frac{b^2}{a^2}}\right) = \int_0^{\pi/2} (a^2 \sin^2 \phi + b^2 \cos^2 \phi)^{-\frac{1}{2}} d\phi.$$

An important consequence of this relation is, that the computation of the complete elliptic integral of the first kind can be very easily performed, since the convergence of the series  $\{a_n\}$  and  $\{b_n\}$



is extremely good (see [2]).

In section 2 we derive by aid of formula (4) the limit expression

$$(5) \quad \lim_{\varepsilon \rightarrow 0} M(1, \varepsilon) \ln \frac{4}{\varepsilon} = \frac{\pi}{2}.$$

Although all results are well-known (see Gauss [1], and Schlesinger [4]), the treatment may be new.

1. The relation between the arithmetico-geometrical mean and the complete elliptic integral of the first kind.

We consider an infinitely thin circular ring, with a uniform distribution of mass of unit density and lying in the plane  $z=0$  of a Cartesian coordinate system  $(x, y, z)$ . When the radius of the ring equals  $R$ , the points of the ring lie at the circle  $x^2 + y^2 = R^2$ . The potential in an arbitrary point  $P(x, y, z)$  is an axially symmetric function and is given by the formula

$$(1.1) \quad u(r, z) = \int_0^{2\pi} (R^2 + r^2 + z^2 - 2Rr \cos \phi)^{-\frac{1}{2}} d\phi,$$

with  $r = (x^2 + y^2)^{\frac{1}{2}}$ .

In particular, for the potential at points of the plane  $z=0$  we obtain after some trivial substitutions

$$(1.2) \quad u(r, 0) = 4 \int_0^{\pi/2} \{(R+r)^2 - 4rR \sin^2 \phi\}^{-\frac{1}{2}} d\phi.$$

The potential in  $P(x, y, z)$  may be obtained in an alternative way by using the well-known formula:

$$f(r, z) = \frac{1}{\pi} \int_0^\pi f(0, z + ir \cos \phi) d\phi$$

valid for axially symmetric harmonic functions (see Whittaker and Watson [3] p. 399).

Therefore we have also:

$$(1.3) \quad u(r,z) = 2 \int_0^{\pi} \{R^2 + (z + ir \cos \phi)^2\}^{-\frac{1}{2}} d\phi$$

Taking the special case  $z=0$ ,  $r < R$ , we obtain

$$(1.4) \quad u(r,0) = 4 \int_0^{\pi/2} \{R^2 - r^2 \cos^2 \phi\}^{-\frac{1}{2}} d\phi.$$

The results (1.2) and (1.4) are of course identical for  $r < R$ , so that an interesting identity is obtained.

By performing the substitutions

$$\alpha = R, \quad \beta = r,$$

$$\alpha_1 = \frac{\alpha + \beta}{2} \text{ and } \beta_1 = \sqrt{\alpha \cdot \beta},$$

we may write this identity in the form:

$$(1.5) \quad \int_0^{\pi/2} (\alpha^2 - \beta^2 \sin^2 \phi)^{-\frac{1}{2}} d\phi = \frac{1}{2} \int_0^{\pi/2} (\alpha_1^2 - \beta_1^2 \sin^2 \phi)^{-\frac{1}{2}} d\phi.$$

Applying the substitutions

$$a = R + r, \quad b = R - r,$$

$$a_1 = \frac{a + b}{2} \text{ and } b_1 = \sqrt{a \cdot b},$$

we obtain a modification of the relation (1.5), namely:

$$(1.6) \quad \int_0^{\pi/2} (a^2 \sin^2 \phi + b^2 \cos^2 \phi)^{-\frac{1}{2}} d\phi = \int_0^{\pi/2} (a_1^2 \sin^2 \phi + b_1^2 \cos^2 \phi)^{-\frac{1}{2}} d\phi.$$

From (1.6) it follows immediately

$$(1.7) \quad \int_0^{\pi/2} (a^2 \sin^2 \phi + b^2 \cos^2 \phi)^{-\frac{1}{2}} d\phi = \lim_{n \rightarrow \infty} \int_0^{\pi/2} (a_n^2 \sin^2 \phi + b_n^2 \cos^2 \phi)^{-\frac{1}{2}} d\phi.$$

It is easily seen that the left-hand side of (1.7) equals

$\frac{1}{a} K(\sqrt{1 - \frac{b^2}{a^2}})$ , whereas the right-hand side equals  $\frac{\pi}{2M(a,b)}$  and hence

$$(1.8) \quad \frac{\pi}{2M(a,b)} = \frac{1}{a} K(\sqrt{1 - \frac{b^2}{a^2}}) = \int_0^{\pi/2} (a^2 \sin^2 \phi + b^2 \cos^2 \phi)^{-\frac{1}{2}} d\phi.$$

## 2. The limit expression for $M(1, \varepsilon)$

By aid of formula (1.8) we have for any  $\varepsilon$ , with  $0 < \varepsilon < 1$ ,

$$\begin{aligned} \frac{1}{M(1, \varepsilon)} &= \frac{1}{2\pi} \int_0^{2\pi} (\sin^2 \phi + \varepsilon^2 \cos^2 \phi)^{-\frac{1}{2}} d\phi = \\ &= \frac{1}{\pi} \int_0^{2\pi} e^{i\phi} [ \{ (1+\varepsilon)e^{2i\phi} - (1-\varepsilon) \} \{ -(1-\varepsilon)e^{2i\phi} + (1+\varepsilon) \} ]^{-\frac{1}{2}} d\phi. \end{aligned}$$

Substituting  $z = e^{i\phi}$ , we obtain

$$(2.1) \quad \frac{1}{M(1, \varepsilon)} = \frac{1}{\pi \sqrt{1-\varepsilon}^2} \oint_C \{ (z^2 - \frac{1-\varepsilon}{1+\varepsilon}) (z^2 - \frac{1+\varepsilon}{1-\varepsilon}) \}^{-\frac{1}{2}} dz$$

where the integration should be performed in the positive sense along the contour  $C$ .  $C$  is the unit circle around the origin of the complex plane, which has cuts as shown in fig. 1,

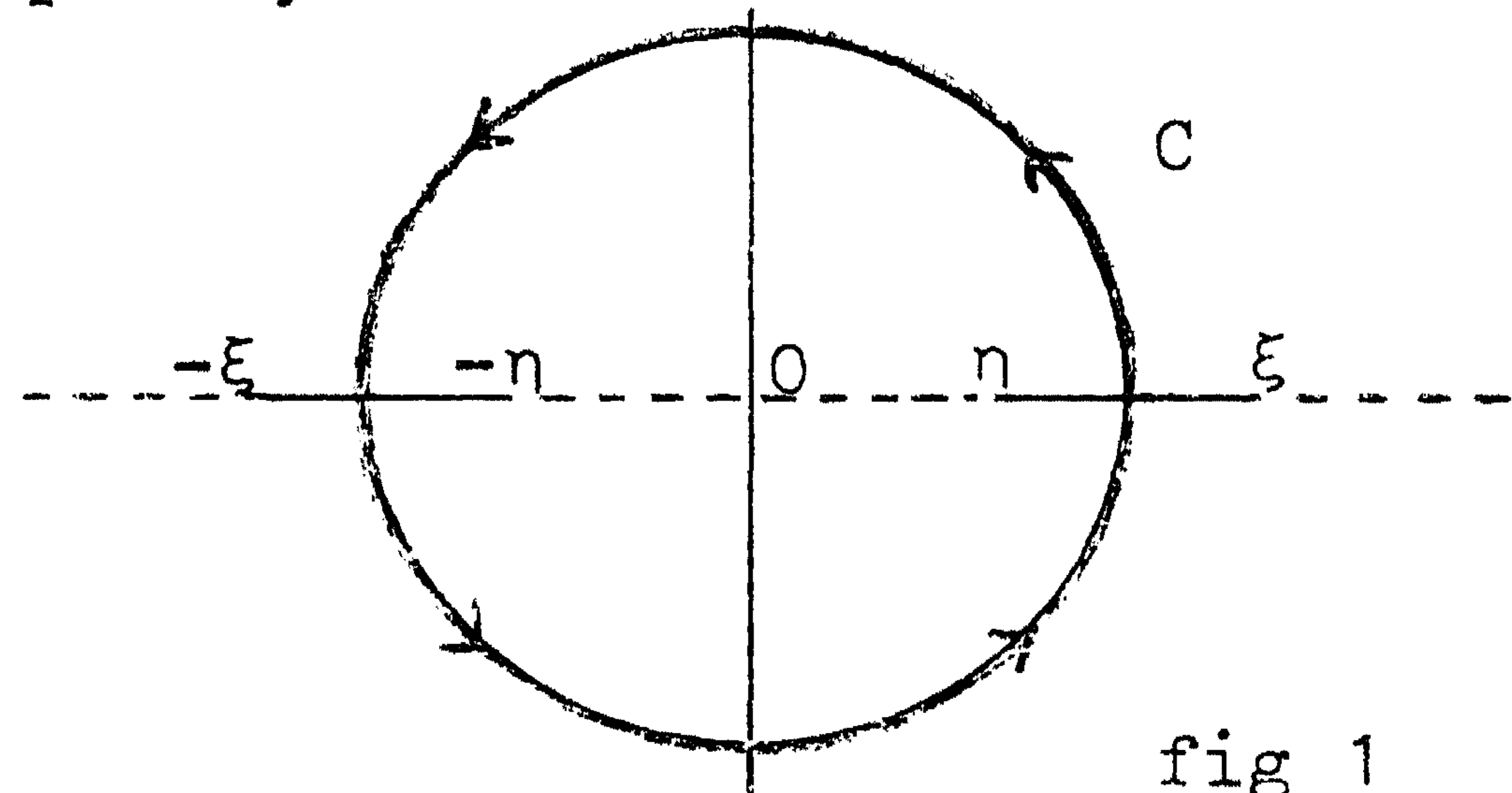


fig 1

where  $\xi = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}$  and  $\eta = \sqrt{\frac{1-\varepsilon}{1+\varepsilon}}$ .

We deform this contour into the contour  $L$ , which consists of the straight lines  $L_1$  and  $L_2$ , parallel to the imaginary axis and intersecting the real axis in the points  $z = -(\xi + \eta)/2$  and  $z = (\xi + \eta)/2$  resp.

This contour is shown in fig. 2.

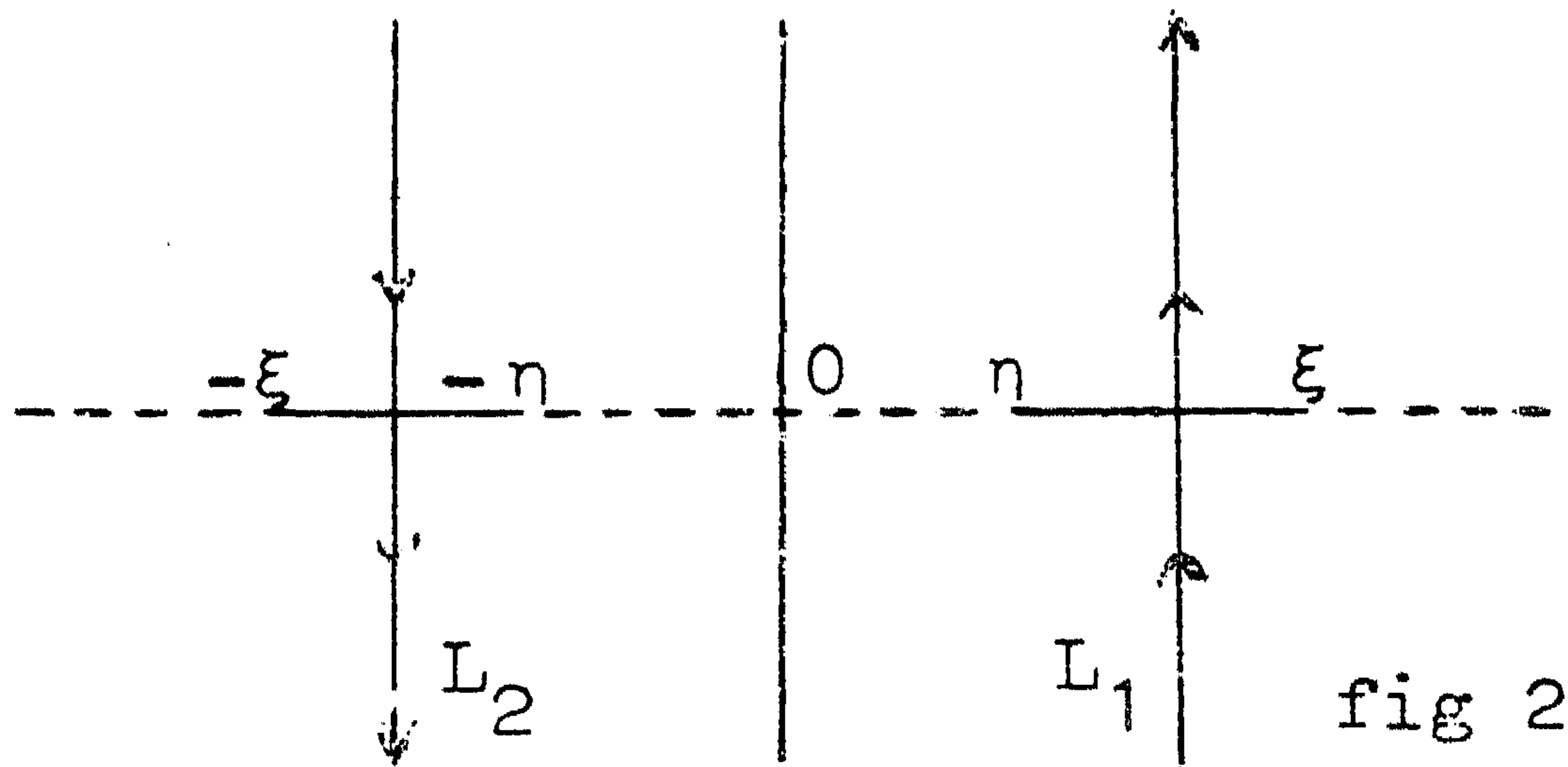


fig 2



After this deformation we may write instead of (2.1):

$$\begin{aligned}
 \frac{\pi\sqrt{1-\varepsilon^2}}{M(1,\varepsilon)} &= \left\{ \int_{-i\infty+\frac{\xi+\eta}{2}}^{+i\infty+\frac{\xi+\eta}{2}} - \int_{-i\infty-\frac{\xi+\eta}{2}}^{+i\infty-\frac{\xi+\eta}{2}} \right\} \{(z^2-\eta^2)(z^2-\xi^2)\}^{-\frac{1}{2}} dz = \\
 &= \int_{-i\infty}^{+i\infty} \{(z+\frac{\xi+\eta}{2})^2-\eta^2\}^{-\frac{1}{2}} \{(z+\frac{\xi+\eta}{2})^2-\xi^2\}^{-\frac{1}{2}} dz - \\
 &- \int_{-i\infty}^{+i\infty} \{(z-\frac{\xi+\eta}{2})^2-\eta^2\}^{-\frac{1}{2}} \{(z-\frac{\xi+\eta}{2})^2-\xi^2\}^{-\frac{1}{2}} dz = \\
 &= \int_{-i\infty}^{+i\infty} (z+\frac{\xi+3\eta}{2})^{-\frac{1}{2}} (z+\frac{\xi-\eta}{2})^{-\frac{1}{2}} (z+\frac{3\xi+\eta}{2})^{-\frac{1}{2}} (z-\frac{\xi-\eta}{2})^{-\frac{1}{2}} dz - \\
 &- \int_{-i\infty}^{+i\infty} (z-\frac{\xi+3\eta}{2})^{-\frac{1}{2}} (z-\frac{\xi-\eta}{2})^{-\frac{1}{2}} (z-\frac{3\xi+\eta}{2})^{-\frac{1}{2}} (z+\frac{\xi-\eta}{2})^{-\frac{1}{2}} dz .
 \end{aligned}$$

Putting now:

$$\frac{\xi-\eta}{2} = \frac{\varepsilon}{\sqrt{1-\varepsilon^2}} = \gamma,$$

$$\frac{\xi+3\eta}{2} = \frac{2-\varepsilon}{\sqrt{1-\varepsilon^2}} = \alpha,$$

$$\frac{3\xi+\eta}{2} = \frac{2+\varepsilon}{\sqrt{1-\varepsilon^2}} = \beta,$$

we obtain:

$$\begin{aligned}
 \frac{\pi\sqrt{1-\varepsilon^2}}{M(1,\varepsilon)} &= \int_{-i\infty}^{+i\infty} (z^2-\gamma^2)^{-\frac{1}{2}} (z+\alpha)^{-\frac{1}{2}} (z+\beta)^{-\frac{1}{2}} dz - \\
 &- \int_{-i\infty}^{+i\infty} (z^2-\gamma^2)^{-\frac{1}{2}} (z-\alpha)^{-\frac{1}{2}} (z-\beta)^{-\frac{1}{2}} dz = \\
 &= \int_{-\infty}^{+\infty} \left[ \{(i\gamma \operatorname{sh} u + \alpha)(i\gamma \operatorname{sh} u + \beta)\}^{-\frac{1}{2}} - \{(i\gamma \operatorname{sh} u - \alpha)(i\gamma \operatorname{sh} u - \beta)\}^{-\frac{1}{2}} \right] du .
 \end{aligned}$$

For small values of  $\varepsilon$  we have  $\alpha = \beta + O(\varepsilon)$  and so we may write

$$\frac{\pi\sqrt{1-\varepsilon^2}}{M(1,\varepsilon)} = \int_{-\infty}^{+\infty} \{ (i\gamma \operatorname{sh} u + \alpha)^{-1} - (i\gamma \operatorname{sh} u - \alpha)^{-1} \} du + r(\varepsilon),$$

with  $r(\varepsilon) \rightarrow 0$  for  $\varepsilon \rightarrow 0$ .

Completing the reduction we obtain finally,

$$\frac{\pi\sqrt{1-\varepsilon^2}}{M(1,\varepsilon)} = 2\alpha \int_{-\infty}^{+\infty} \frac{du}{\gamma^2 \operatorname{sh}^2 u + \alpha^2} + r(\varepsilon) = \frac{2}{\sqrt{\alpha^2 - \gamma^2}} \ln \frac{\{\alpha + \sqrt{\alpha^2 - \gamma^2}\}^2}{\gamma^2} + r(\varepsilon),$$

or

$$(2.2) \quad \frac{\pi}{2M(1,\varepsilon)} = \ln \frac{4}{\varepsilon} + r_1(\varepsilon), \text{ where } r_1(\varepsilon) \rightarrow 0 \text{ for } \varepsilon \rightarrow 0.$$

Hence we arrive at the desired result:

$$(2.3) \quad \lim_{\varepsilon \rightarrow 0} M(1,\varepsilon) \ln \frac{4}{\varepsilon} = \frac{\pi}{2}.$$

### References

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